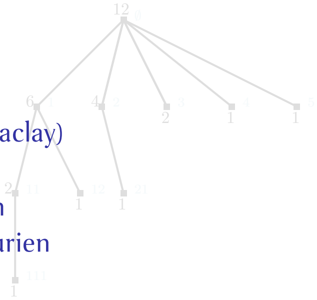
Loop $O(n)$ model on random quadrangulations: the cascade of loop perimeters



Pascal Maillard

(Université Paris-Sud / Paris-Saclay)

based on joint work with

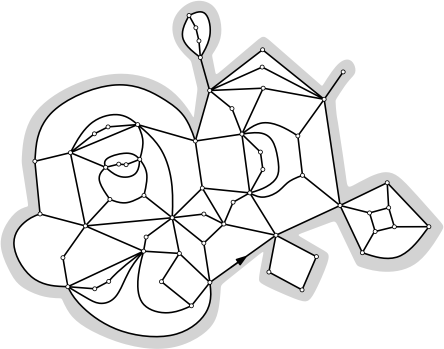
Linxiao Chen and Nicolas Curien

Cargèse, 23 September 2016

Model and results

Definitions

A *bipartite map with a boundary* is a rooted bipartite map in which the face on the right of the root edge is called the *external face*, and the other faces called *internal faces*.
A *quadrangulation with a boundary* is a bipartite map with a boundary whose internal faces are all quadrangles.



Remark

The boundary is not necessarily simple.

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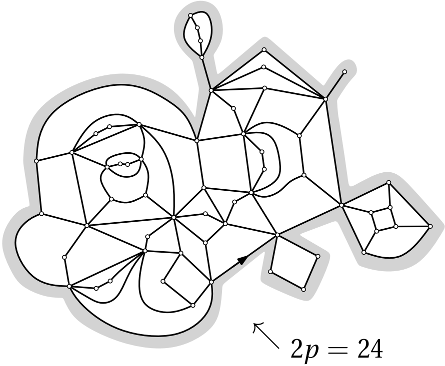

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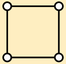
We denote by $2p$ the *perimeter* of the map (i.e. degree of the external face).

Loop $O(n)$ model on quadrangulations

A *loop configuration* on a quadrangulation with boundary $\mathfrak{q}$ is a collection of *disjoint simple closed paths* on the dual of $\mathfrak{q}$ which do not visit the external face. We restrict ourselves to the so-called *rigid* loops, i.e. such that every internal face is of type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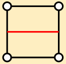

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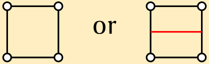

$$\mathcal{O}_p = \left\{ (\mathfrak{q}, \boldsymbol{\ell}) \middle| \begin{array}{c} \mathfrak{q} \text{ is a quadrangulation with a boundary of length } 2p, \\ \boldsymbol{\ell} \text{ is a rigid loop configuration on } \mathfrak{q}. \end{array} \right\}$$

For $n \in (0, 2)$ and $g, h > 0$, let

$$F_p(n; g, h) = \sum_{(\mathfrak{q}, \boldsymbol{\ell}) \in \mathcal{O}_p} g^{\# \square} h^{\# \boxminus} n^{\# \circlearrowleft}$$

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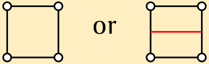

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A triple $(n; g, h)$ is *admissible* if $F_p(n; g, h) < \infty$. (This is independent of p).

Loop $O(n)$ model on quadrangulations

Definition

Fix $p > 0$. For each admissible triple $(n; g, h)$, we define a probability distribution on $\mathcal{O}_p$ by

$$\mathbb{P}^{(p)}_{n;g,h}((\mathfrak{q}, \boldsymbol{\ell})) = \frac{g^{\#} h^{\#} n^{\#}}{F_p(n; g, h)}$$

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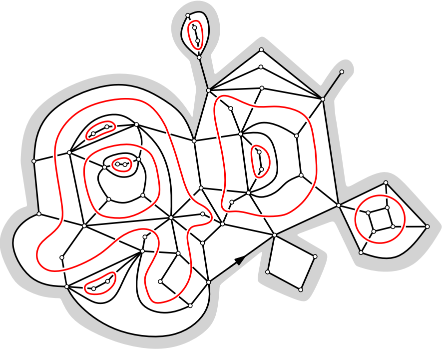

$$\rightsquigarrow \quad \mathbb{P}^{(12)}_{n;g,h}(\,\cdot\,) = \frac{g^8\, h^{38}\, n^9}{F_{12}(n; g, h)}$$

Theorem (Borot, Bouttier, Guitter '12)

For all admissible $(n; g, h)$, there exist $\kappa(n; g, h)$ and $\alpha(n; g, h)$ such that

$$F_p(n; g, h) \underset{p\to\infty}{\sim} C\,\kappa^{-p}\,p^{-\alpha-1/2}$$

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For each $n \in (0, 2)$, there are *four* possible values of α

subcritical: $\alpha = 1$ generic critical: $\alpha = 2$

non-generic critical

dense phase: $\alpha = \frac{3}{2} - \frac{1}{\pi}\arccos(n/2) \quad \in (1, 3/2)$

dilute phase: $\alpha = \frac{3}{2} + \frac{1}{\pi}\arccos(n/2) \quad \in (3/2, 2)$

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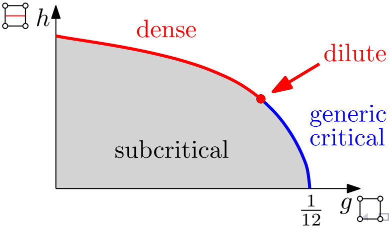
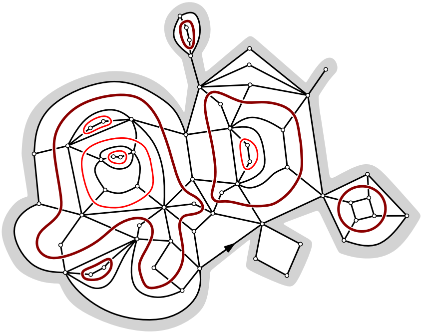
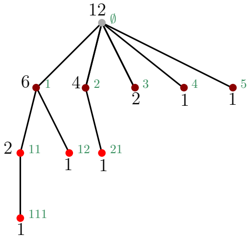


The perimeter cascade of loops

We focus on the *hierarchical structure* of the loops, which we represent by a tree labeled by the *half-perimeters* of the loops.





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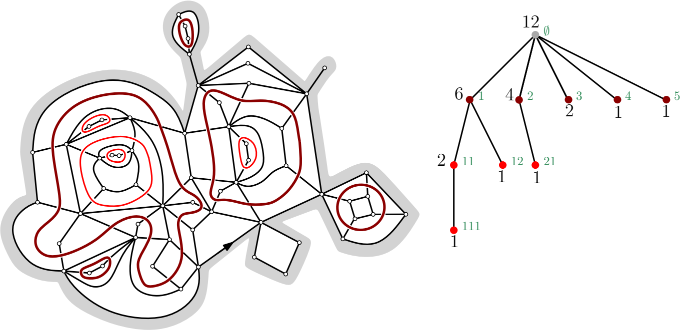


We complete the tree by vertices of label 0. This gives a random process $(\chi^{(p)}(u))_{u\in\mathcal{U}}$ indexed by the *Ulam tree* $\mathcal{U} = \bigcup_{n\geq 0}(\mathbb{N}^*)^n$. We call this process the *(half-)perimeter cascade* of the rigid loop $O(n)$ model on quadrangulations.

Main results

Theorem (CCM 2016+)

Let $(\chi^{(p)}(u))_{u\in\mathcal{U}}$ be the previously defined perimeter cascade. Then, we have the following convergence in distribution in $\ell^\infty(\mathcal{U})$:

$$\left(p^{-1}\chi^{(p)}(u)\right)_{u\in\mathcal{U}} \overset{p\to\infty}{\Longrightarrow} (Z_\alpha(u))_{u\in\mathcal{U}},$$

where $Z_\alpha = (Z_\alpha(u))_{u\in\mathcal{U}}$ is a multiplicative cascade to be defined later.

Related results

- Borot, Bouttier, Duplantier '16: Number of loops surrounding a marked vertex.
- Common belief: map + $O(n)$ loops $\leftrightarrow$ Liouville quantum gravity + conformal loop ensemble (more on this later).
- huge literature on random planar maps with statistical mechanics model (uniform spanning tree, Potts model) in different scientific fields (combinatorics, probability, physics)
- Random planar map without statistical mechanics model, endowed with graph metric: limiting metric space is *Brownian Map* (Miermont, Le Gall '13)

Multiplicative cascades

Multiplicative cascades

Definition

A *multiplicative cascade* is a random process $Z = (Z(u))_{u\in\mathcal{U}}$ such that

$$Z(\emptyset) = 1, \quad \forall u \in \mathcal{U},\ i \geq 1 : Z(ui) = Z(u) \cdot \xi(u,i),$$

where $(\boldsymbol{\xi}(u))_{u\in\mathcal{U}} = (\xi(u,i), i \geq 1)_{u\in\mathcal{U}}$ is an i.i.d. family of random vectors in $(\mathbb{R}_+)^{\mathbb{N}^*}$. The law of $\boldsymbol{\xi} = \boldsymbol{\xi}(\emptyset)$ is the *offspring distribution* of the cascade Z.

Remark: $X = \log Z = (\log Z(u))_{u\in\mathcal{U}}$ is a *branching random walk*.

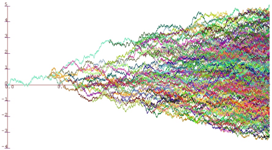
Multiplicative cascades and branching random walks: a short history

Cascades multiplicatives: Mandelbrot, Kahane, Peyrière...

- Motivation: Model of the energy cascade in turbulent fluids
- Studied mostly on d-ary tree (i.e. $\xi_i = 0$ pour $i > d$).
- Multiplicative cascade gives a random measure on the tree boundary, theory mostly studies the multifractal properties of this random measure. Interaction between geometry of the tree and the values of the process $Z(u)$.

Branching random walks: Hammersley, Kingman, Biggins...

- Motivation: Generalisation of the Crump-Mode-Jagers process (branching process with age)
- u: particle, $X(u)$: position of the particle u.
- Theory mostly studies the distribution of the particle positions, ignoring the geometry of the tree. Particular focus on extremal particles.



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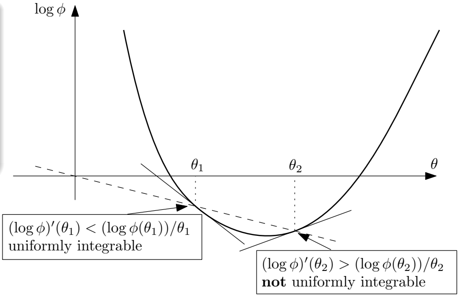
Mellin transform and martigales of multiplicative cascades

Definition (Mellin transform)

$$\phi(\theta) := \mathbb{E}\left[\sum_{i\in\mathbb{N}^*} \xi(i)^\theta\right] \in (-\infty, +\infty]$$

- $\log\phi$ is convex
- $W_n^{(\theta)} := \phi(\theta)^{-n}\sum_{|u|=n} Z(u)^\theta$ is a martingale.



Theorem (Biggins, Lyons)

$(W_n^{(\theta)})_{n\geq 0}$ is uniformly integrable (u.i.) if and only if $\mathbb{E}[W_1^{(\theta)}\log^+ W_1^{(\theta)}] < \infty$ and $(\log\phi)'(\theta) < (\log\phi(\theta))/\theta$

The multiplicative cascade Z_α

- $(\zeta_t)_{t\geq 0}$: α-stable Lévy process without negative jumps, started from 0.
- τ: the hitting time of -1 by ζ.
- $(\Delta\zeta)^\downarrow_\tau$: the jumps of ζ before τ, sorted in $\downarrow$ order.
- $\mathrm{d}\,\nu_\alpha := \frac{1/\tau}{\mathbb{E}[1/\tau]}\mathrm{d}\,\widetilde{\nu}_\alpha$, where $\tilde{\nu}_\alpha$ is the law of $(\Delta\zeta)^\downarrow_\tau$

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Theorem (CCM 2016+)

Let $(\chi^{(p)}(u))_{u\in\mathcal{U}}$ be the perimeter cascade of the rigid loop $O(n)$ model on quadrangulations. Then we have the convergence in distribution in $\ell^\infty(\mathcal{U})$:

$$\left(p^{-1}\chi^{(p)}(u)\right)_{u\in\mathcal{U}} \overset{p\to\infty}{\Longrightarrow} (Z_\alpha(u))_{u\in\mathcal{U}},$$

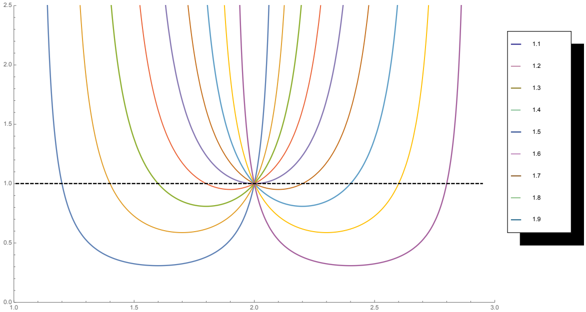
where $(Z_\alpha(u))_{u\in\mathcal{U}}$ is a multiplicative cascade of offspring distribution ν_α.

Properties of Z_α

Theorem (CCM 2016+)

The Mellin transform of the multiplicative cascade Z_α is

$$\phi_\alpha(\theta) = \frac{\sin(\pi(2-\alpha))}{\sin(\pi(\theta-\alpha))} \quad \textit{pour } \theta \in (\alpha, \alpha+1) \quad \textit{and} \quad \phi_\alpha(\theta) = \infty \textit{ otherwise.}$$



Intrinsic martingales

If $\phi_\alpha(\theta) = 1$, then

$$W_n^{(\theta)} = \sum_{|u|=n} Z_\alpha(u)^\theta$$

is called an intrinsic martingale. For $\alpha \neq 3/2$, there are two intrinsic martingales with $\theta = 2$ and $\theta = 2\alpha - 1$. It follows from Biggins' theorem that

- if $\alpha \in (3/2, 2)$ (dilute phase), then $2 < 2\alpha - 1$, hence $W^{(2)}$ is u.i., whereas $W^{(2\alpha-1)}$ is not,
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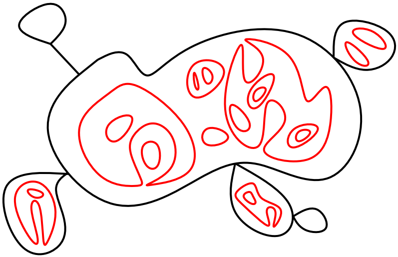
This suggests the following for the volume Vol_p of the random quandragulation with perimeter p:

Volume scaling

dilute phase: Vol_p / p^2 converges in law to $W_\infty^{(2)}$ as $p \to \infty$
dense phase: $\mathrm{Vol}_p / p^{2\alpha-1}$ converges in law to $W_\infty^{(2\alpha-1)}$ as $p \to \infty$.

Proofs

The gasket decomposition [Borot, Bouttier, Guitter '12]



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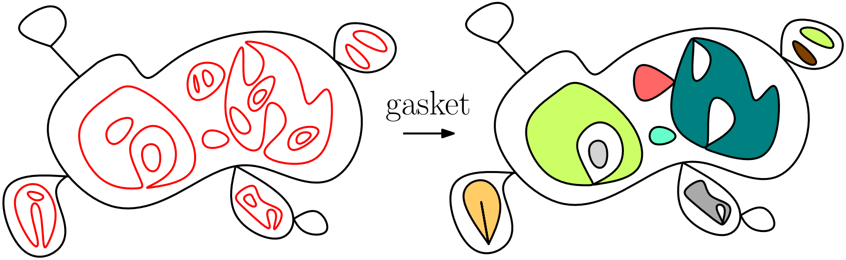

gasket: a bipartite map

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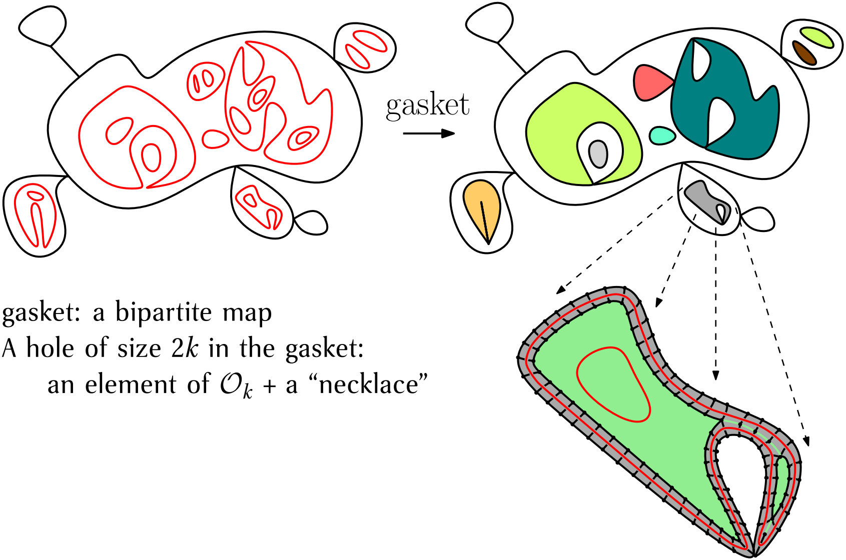

gasket: a bipartite map

A hole of size $2k$ in the gasket:

an element of $\mathcal{O}_k$ + a "necklace"

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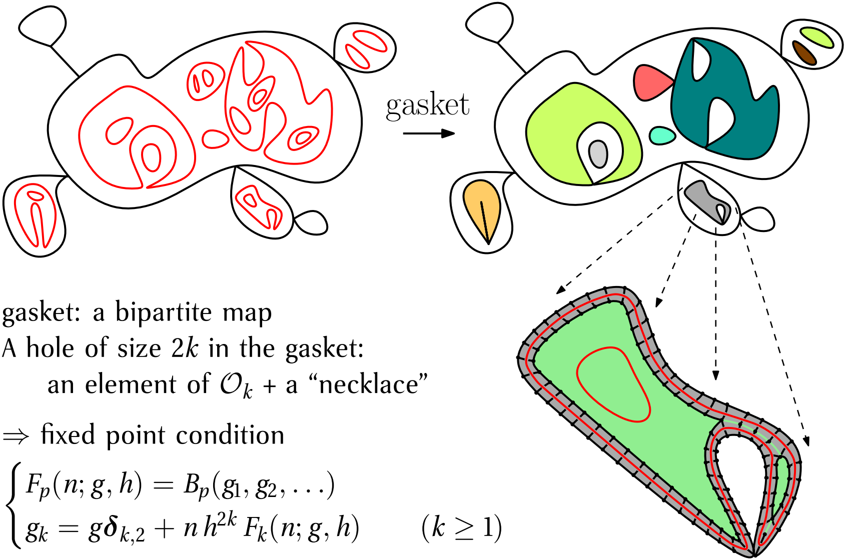

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$\Rightarrow$ fixed point condition

$$\begin{cases} F_p(n; g, h) = B_p(g_1, g_2, \dots) \\ g_k = g\delta_{k,2} + n\, h^{2k} F_k(n; g, h) \qquad (k \geq 1) \end{cases}$$

The gasket decomposition

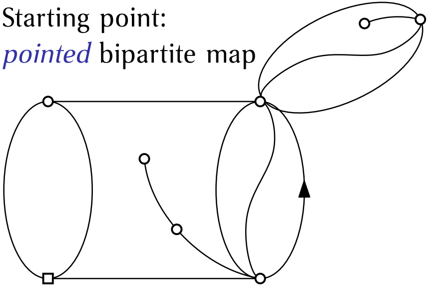
The gasket decomposition

A (head) gasket.

Encoding the gasket: the BDG and JS bijections

[Bouttier, Di Francesco, Guitter '04, Janson, Stefánsson '15]

Starting point:
pointed bipartite map



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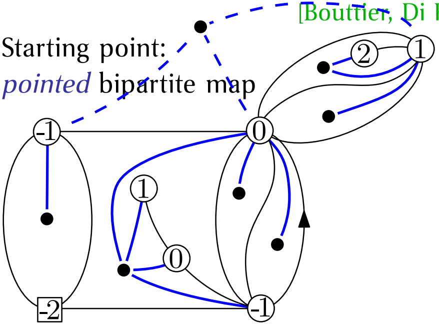

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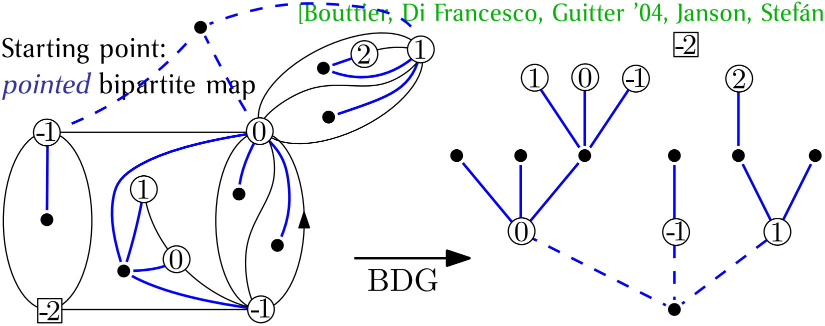

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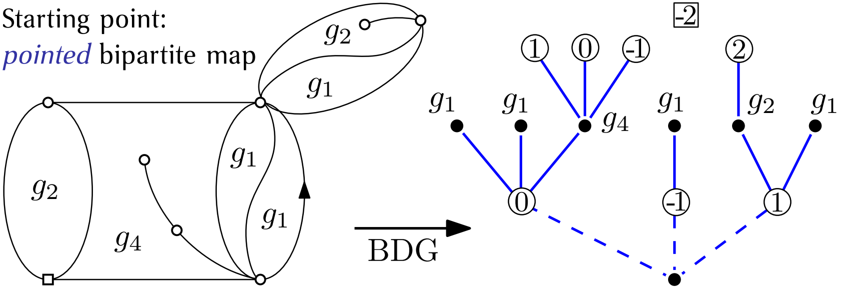

$g_k \rightsquigarrow$ face of degree $2k$

$\xrightarrow{BDG}$ $g_k \rightsquigarrow$ $\bullet$ of degree k

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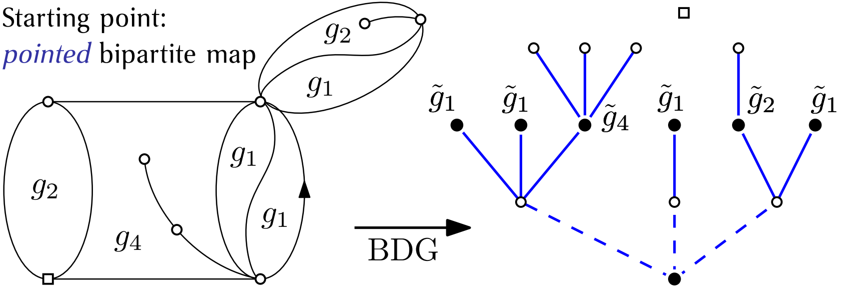

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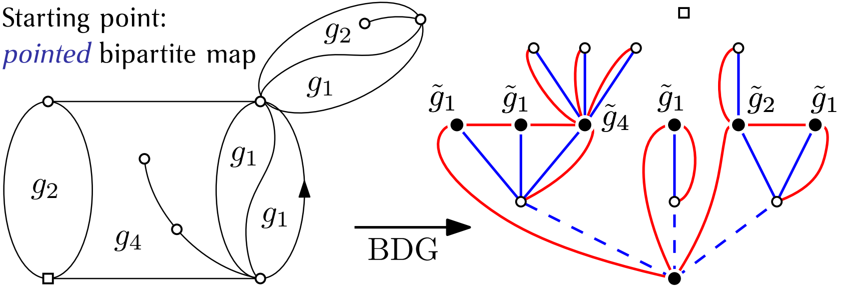

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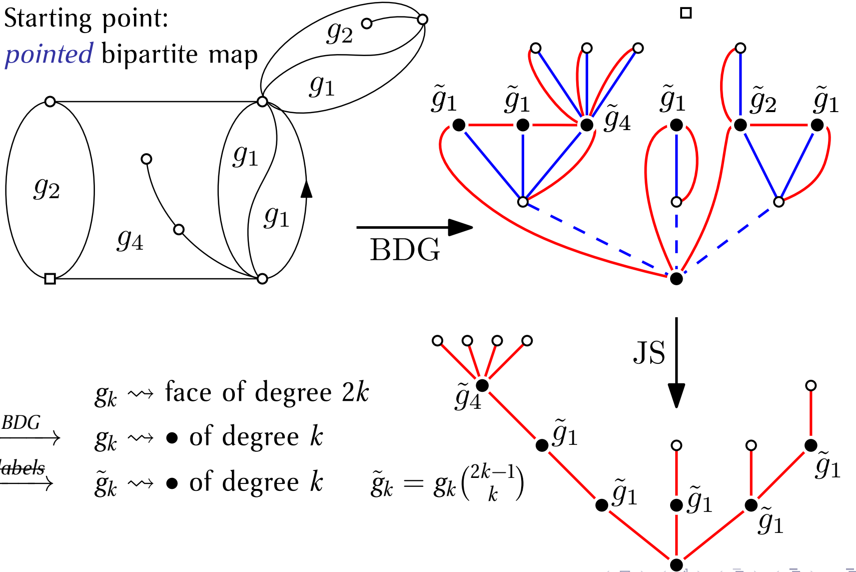

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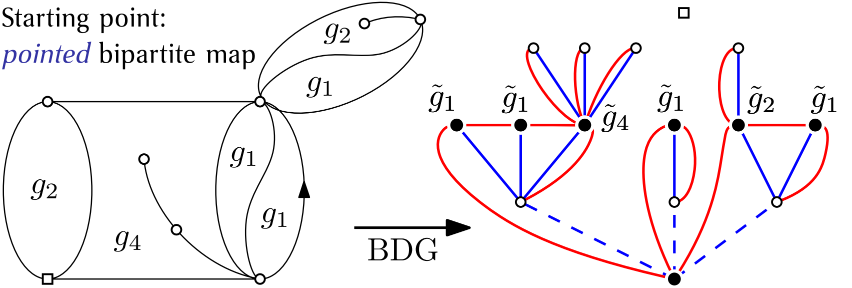

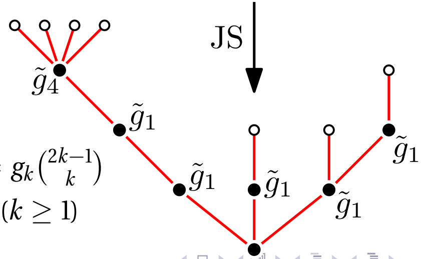

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$\xrightarrow{JS}$ $\tilde{g}_k \rightsquigarrow \bullet$ with k descendants ($k \geq 1$)

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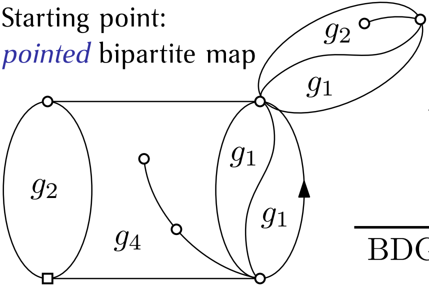

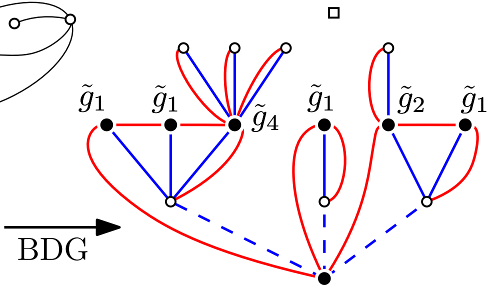

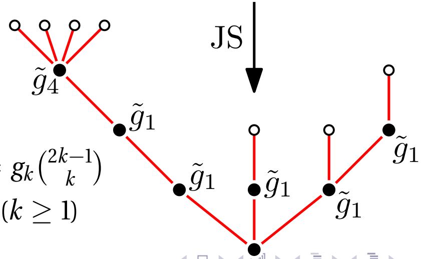

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$\xrightarrow{BDG}$ $g_k \rightsquigarrow \bullet$ of degree k

$\xrightarrow{\cancel{labels}}$ $\tilde{g}_k \rightsquigarrow \bullet$ of degree k $\quad \tilde{g}_k = g_k \binom{2k-1}{k}$

$\xrightarrow{JS}$ $\tilde{g}_k \rightsquigarrow \bullet$ with k descendants ($k \geq 1$)

($1 \rightsquigarrow \circ$ with 0 descendant)

Encoding the gasket: the BDG and JS bijections

pointed bipartite maps under the Boltzmann distribution

$$\mathbb{P}^{\bullet}_{p,\mathbf{g}}(M = \mathfrak{m}^{\bullet}) = \frac{\prod_{k=1}^{\infty} g_k^{f_k(\mathfrak{m}^{\bullet})}}{B^{\bullet}_p(\mathbf{g})}$$

$\xrightarrow[\text{JS}]{\text{BDG}}$

Galton-Watson tree of offspring distribution

$$\mu_{\mathtt{JS}}(k) = \tilde{g}_k \kappa^{k-1} \sim Ck^{-\alpha}$$

face of degree $2k$	$\longrightarrow$	internal vertex with k children
vertices	$\longrightarrow$	leaves

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The BDG-JS bijection applies naturally to *pointed* bipartite maps. To recover a *non-pointed* Boltzmann map, we need to bias the law of the Galton-Watson tree by $1/\{$its number of leaves$\}$.

$$\mathbb{E}_{p,\mathbf{g}}[F(M)] = \frac{\mathbb{E}^\bullet_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}}F(M)\right]}{\mathbb{E}^\bullet_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}}\right]} = \frac{\mathbb{E}_{\mathrm{GW}}\left[\frac{1}{\#\text{leaf}}F(T)\right]}{\mathbb{E}_{\mathrm{GW}}\left[\frac{1}{\#\text{leaf}}\right]}$$

Encoding the gasket: scaling limit of the hole sizes

Conclusion

Let $(\chi^{(p)}(i))_{i\geq 1}$ be the half-degrees of faces of the gasket, sorted in $\downarrow$ order and completed with zeros. Then for all bounded functions F,

$$\mathbb{E}[F(\chi^{(p)}(i))] = \frac{\mathbb{E}\left[\frac{1}{\#\{i\leq T_p : X_i = -1\}} F((X_i+1)^{\downarrow}_{T_p})\right]}{\mathbb{E}\left[\frac{1}{\#\{i\leq T_p : X_i = -1\}}\right]}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ is a random walk with step distribution $\mu(k) = \mu_{\mathrm{JS}}(k+1)$ $(k \geq -1)$ and T_p its hitting time of $-p$.

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When p is large, $\#\{i \leq T_p : X_i = -1\} \approx \mu(-1) T_p$.

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Proposition

$(p^{-1}\chi^{(p)}(i))_{i\geq 1} \underset{p\to\infty}{\Longrightarrow} \nu_\alpha$ as $p \to \infty$ in the sense of finite dimensional marginals.

An identity on random walks

Theorem (CCM)

Let $S_n = X_1 + \cdots + X_n$ be a random walk with steps $X_i \in \{-1, 0, 1, \cdots\}$. Let T_p be its hitting time of $-p$. Then, for all $f : \mathbb{Z} \to \mathbb{R}_+$ and all $p \geq 2$,

$$\mathbb{E}\left[\frac{1}{T_p - 1}\sum_{i=1}^{T_p} f(X_i)\right] = \mathbb{E}\left[f(X_1)\frac{p}{p+X_1}\right].$$

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Theorem (CCM)

Let $(\eta_t)_{t\geq 0}$ *be a Lévy process without negative jumps and of Lévy measure* π*.*
Let τ *be its hitting time at* -1*. Then, for all measurable* $f : \mathbb{R}_+^* \to \mathbb{R}_+$

$$\mathbb{E}\left[\frac{1}{\tau}\sum_{t\leq\tau} f(\Delta\eta_t)\right] = \int f(x)\frac{1}{1+x}\pi(\mathrm{d}x).$$

Proof of the discrete identity

Kemperman's formula (/cyclic lemma /ballot theorem . . .)

If the F is invariant under cyclic permutation of its arguments, then

$$\mathbb{E}\left[F(X_1,\cdots,X_n)\mathbf{1}_{\{T_p=n\}}\right] = \frac{p}{n}\mathbb{E}\left[F(X_1,\cdots,X_n)\mathbf{1}_{\{S_n=-p\}}\right]$$

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Proof.

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A_n := \mathbb{E}\left[\sum_{i=1}^n f(X_i)\mathbf{1}_{\{T_p=n\}}\right] &= \frac{p}{n}\,\mathbb{E}\left[\sum_{i=1}^n f(X_i)\mathbf{1}_{\{S_n=-p\}}\right] && \text{by Kemperman's formula}\\
&= p\,\mathbb{E}\left[f(X_1)\mathbf{1}_{\{S_n=-p\}}\right] && \text{by cyclic symmetry}\\
&= p\,\mathbb{E}\left[f(X_1)\mathbf{1}_{\{\tilde{S}_{n-1}=-p-X_1\}}\right] && \text{by Markov property}\\
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\end{aligned}$$

For $p \geq 2$ we have always $T_p \geq 2$, hence

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{T_p-1}\sum_{i=1}^{T_p} f(X_i)\right] = \sum_{n=2}^{\infty}\frac{A_n}{n-1} &= p\sum_{n=2}^{\infty}\mathbb{E}\left[f(X_1)\frac{1}{p+X_1}\mathbf{1}_{\{\tilde{T}_{p+X_1}=n-1\}}\right] \\
&= \mathbb{E}\left[f(X_1)\frac{p}{p+X_1}\right].
\end{aligned}$$

Consequences of the identities

- The Mellin transform of the continuous cascade Z_α: for $\theta \in (\alpha, \alpha+1)$,

$$\frac{\mathbb{E}\left[\frac{1}{\tau}\sum_{t\le\tau}(\Delta\eta_t)^\theta\right]}{\mathbb{E}\left[\frac{1}{\tau}\right]} = \frac{\int \frac{x^\theta}{1+x}\pi(\mathrm{d}x)}{\int \frac{1}{1+x}\pi(\mathrm{d}x)} = \frac{\sin(\pi(2-\alpha))}{\sin(\pi(\theta-\alpha))}$$

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- Convergence of moments of the offspring distribution

$$\mathbb{E}\left[\sum_{i=1}^\infty \left(p^{-1}\chi^{(p)}(i)\right)^\theta\right] \xrightarrow[p\to\infty]{} \mathbb{E}\left[\sum_{i=1}^\infty (Z_\alpha(i))^\theta\right]$$

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- For all $k \in \mathbb{N}$: convergence in $\ell^\infty(\mathcal{U}_k)$ of the perimeter cascade ($\mathcal{U}_k$: first k generations of the Ulam tree).
- Convergence in $\ell^\infty(\mathcal{U})$: NOT a consequence, obtained by other methods (martingale inequalities and exact bounds on volume of random quadrangulations with small perimeter).

Relation with results on CLE

Map + $O(n)$ $\leftrightarrow$ LQG + CLE

Common belief: $\exists$ embedding of planar maps to unit disk $\mathbb{D}$ (uniformization, circle packing...), such that

volume measure of random planar map + $O(n)$ loops $\to$ LQG$_\gamma$ + CLE$_\kappa$

Parameters related by

$$\alpha - \frac{3}{2} = \pm\frac{1}{\pi}\arccos(n/2) = \frac{4}{\kappa} - 1, \quad \gamma = \sqrt{\min(\kappa, 16/\kappa)}$$

Let's focus on the dilute phase:

$$\alpha > 3/2, \quad \kappa < 4, \quad \gamma = \sqrt{\kappa}$$

Then we saw before: $\mathrm{Vol}_p \sim p^2$.

Nb of loops around small balls in random quadrangulation

For $\delta > 0$ (small), consider the set $\mathcal{L}_\delta$ of vertices u in the Ulam tree such that $Z_\alpha(u)^2 < \delta$ and $Z_\alpha(v)^2 \geq \delta$ for all $v \prec u$. Define

$$W^{(\theta),\delta} = \sum_{u \in \mathcal{L}_\delta} \varphi_\alpha(\theta)^{-|u|} Z_\alpha(u)^\theta.$$

Then since $(W_n^{(\theta)})_{n \geq 0}$ is u.i.,

$$1 = \mathbb{E}[W^{(\theta),\delta}] \approx \delta^{\theta/2} \mathbb{E}[\sum_{u \in \mathcal{L}_\delta} \varphi_\alpha(\theta)^{-|u|}].$$

Suggests: if we partition the vertices of the quadrangulation into metric balls $B_\delta(v)$ of volume δ and denote by $N_\delta(v)$ the number of vertices surrounding the ball $B_\delta(v)$, then (cf Borot, Bouttier, Duplantier '16)

$$\mathbb{E}[\sum_v \phi_\alpha(\theta)^{-N_\delta(v)}] \approx \delta^{-\theta/2}.$$

Nb of loops around small quantum balls, $\mathrm{LQG}_{\sqrt{\kappa}}$ + CLE_κ

$\widetilde{N}_r$ = number of CLE_κ loops surrounding Euclidean ball of radius $r > 0$.
Then (Schramm, Sheffield, Wilson '09, Miller, Sheffield, Watson '16)

$$\mathbb{E}[\psi_\kappa(\widetilde{\theta})^{-\widetilde{N}_r}] \approx r^{-\widetilde{\theta}}, \quad \text{where}$$

$$\psi_\kappa(\widetilde{\theta}) = \frac{-\cos(4\pi/\kappa)}{\cos(\pi\sqrt{(1-4/\kappa)^2 - 8\widetilde{\theta}/\kappa})} = \phi_\alpha(1 + \frac{4}{\kappa} - \sqrt{(1-4/\kappa)^2 - 8\widetilde{\theta}/\kappa}).$$

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Explanation (cf BBD16 for similar derivation): partition space into squares of quantum volume $\approx \delta$. $N(S)$ = number of CLE loops surrounding square S. Then,

$$\mathbb{E}[\sum_S \psi_\kappa(\widetilde{\theta})^{-\widetilde{N}(S)}] \approx \delta^{\frac{1}{2}(-1-\frac{4}{\kappa}+\sqrt{(1-4/\kappa)^2-8\widetilde{\theta}/\kappa})}.$$

Comparison with quandragulations: $\theta = 1 + \frac{4}{\kappa} - \sqrt{(1-4/\kappa)^2 - 8\widetilde{\theta}/\kappa}$,
$\psi_\kappa(\widetilde{\theta}) = \phi_\alpha(\theta)$.

Thank you for your attention !